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Standard Deviation, Expectation Value, and Uncertainty

In Section 2.5 of *A Student's Guide to the Schrödinger Equation*, Equation 2.61 gives the square of the uncertainty in the measurement of a quantum observable as

$$(\Delta x)^2 = \langle x^2 \rangle - \langle x \rangle^2$$

which is said to be equivalent to the variance of *x*. This makes the uncertainty equivalent to the standard deviation of *x*:

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$$

as stated in Equation 2.62. In this document, you can see where these expressions relating expectation values to variance and standard deviation come from.

Start by recalling that the average or expectation value for a continuous variable *x* is given by

$$\langle x \rangle = \int xP(x)dx$$

in which *P(x)* represents the probability density as a function of x.

Now think about the definition of variance, which is the average value of the square of the difference between *x* and the average value of the distribution:

Variance ≡ average value of $(x - \langle x \rangle)^2$.

But the average value of $(x - \langle x \rangle)^2$ is given by

$$\langle (x - \langle x \rangle)^2 \rangle = \int (x - \langle x \rangle)^2 P(x)dx.$$

Squaring the quantity in parentheses inside the integral gives

$$\langle (x - \langle x \rangle)^2 \rangle = \int [x^2 - 2x\langle x \rangle + \langle x \rangle^2] P(x)dx$$

$$= \int x^2 P(x)dx - \int 2x\langle x\rangle P(x)dx + \int \langle x\rangle^2 P(x)dx$$

or

$$\langle (x - \langle x\rangle)^2\rangle = \int x^2 P(x)dx - 2\langle x\rangle \int xP(x)dx + \langle x\rangle^2 \int P(x)dx$$

(since average quantities are constant, they can be moved outside of the integrals). But the first of these three terms is just the expectation value of x^2:

$$\int x^2 P(x)dx = \langle x^2\rangle$$

and the second term is

$$2\langle x\rangle \int xP(x)dx = 2\langle x\rangle\langle x\rangle = 2\langle x\rangle^2$$

while in the third term $\int P(x)dx = 1$ (since the probability over all *x* must be one), so

$$\langle x\rangle^2 \int P(x)dx = \langle x\rangle^2.$$

This means that

$$\langle (x - \langle x\rangle)^2\rangle = \langle x^2\rangle - 2\langle x\rangle^2 + \langle x\rangle^2 = \langle x^2\rangle - \langle x\rangle^2,$$

which matches Eq. 2.61, so the square of the uncertainty is indeed equivalent to the variance.

Taking the square root of the variance gives the standard deviation (σ), so

$$\sigma = \sqrt{\langle (x - \langle x\rangle)^2\rangle} = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$$

which matches Eq. 2.62, so the uncertainty is equivalent to the standard deviation.